2019-Nov-21 (pre-lecture)

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Outline:

- Difference equations vs recurrence relations
- Generating functions

Last time:

- We defined a first-order difference equation by

$$a_{n+1} = f(n, a_n).$$

This time:

Note that $a_{n+1} = f(n, a_n)$ looks similar to

$$\dot{x} = f(t, x),$$ but not quite.

A more direct analog would be to define a series of finite differences

$$\Delta(a_n) = a_{n+1} - a_n \qquad \approx \qquad \frac{dx}{dt}$$

$$\Delta^2(a_n) = \Delta(a_{n+1}) - \Delta(a_n) \qquad \approx \qquad \frac{d^2x}{dt^2}$$

$$\Delta^3(a_n) = \Delta^2(a_{n+1}) - \Delta^2(a_n) \qquad \vdots$$

$$\vdots$$

$$\Delta^k(a_n) = \Delta^{k-1}(a_{n+1}) - \Delta^{k-1}(a_n)$$

then

$$\Delta^2(a_n) = a_{n+2} - 2a_{n+1} + a_n$$

$$\Delta^3(a_n) = a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n$$

$$\vdots$$

$$\Delta^k(a_n) = \sum_{t=1}^{k} \binom{k}{t} (-1)^t a_{n+k-t}$$

We could define a difference equation as a function of finite differences $\Delta^k(y_n)$, but it is easier to work directly with recurrence relations

$$a_{n+k} = f(n, a_n, a_{n+1}, \ldots, a_{n+k-1})$$

$$a_{n+k} = f(n, a_n, a_{n+1}, \ldots, a_{n+k-1})$$

↑ kth order

Recurrence relations, given initial conditions, define a series of terms $a_0, a_1, a_2, \ldots$

Given a series of real numbers, we can define a **formal power series**

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Sometimes, a formal power series will be a simple function.

e.g. $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n$$

But we can also do manipulations on power series as if they were functions.

- Given $f(x) = \sum_i a_i x^i$, $\quad g(x) = f(\lambda x) = \sum_i \lambda^i a_i x^i$
- Given $f(x) = \sum_i a_i x^i$, $g(x) = \sum_i b_i x^i$, $\quad f(x) + g(x) = \sum_i (a_i + b_i) x^i$
- Given $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $\quad x f(x) = \sum_{i=0}^{\infty} (a_i) x^{i+1} = \sum_{i=1}^{\infty} (a_{i-1}) x^i$
- Given $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $\quad f'(x) = \sum_{i=1}^{\infty} i\, a_i\, x^{i-1} = \sum_{i=0}^{\infty} (i+1)\, a_{i+1}\, x^i$
- If $f(x)$ is the generating function for a_n
  $g(x)$ is the generating function for b_n, then

  $f(x) \cdot g(x)$ is the generating function for $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

ex. $f(x) = 1 + x + x^2 + x^3 + \cdots \qquad \left(\frac{1}{1-x}\right)$

$g(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \qquad (e^x)$

$f(x)g(x) = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \cdots \qquad \left(\frac{e^x}{1-x}\right)$

Can verify by taking the Taylor series of $h(x) = \frac{e^x}{1-x}$ ✓

Can verify by taking the Taylor series of $h(x) = \frac{e^x}{1-x}$

$$h(x) = \frac{e^x}{1-x} \qquad h(0) = 1$$

$$h'(x) = \frac{e^x}{(1-x)^2} + h(x) \qquad h'(0) = 2$$

$$h''(x) = \frac{2e^x}{(1-x)^3} + \frac{e^x}{(1-x)^2} + h'(x) \qquad h''(0) = 5 \qquad \frac{5}{2}$$

$$h'''(x) = \frac{6e^x}{(1-x)^4} + \frac{4e^x}{(1-x)^3} + \frac{e^x}{(1-x)^2} + h''(x) \qquad h'''(0) = 16 \qquad \frac{16}{6} = \frac{8}{3}$$

Back to solving difference equations / recurrence relations

Ex. $a_n = 3a_{n-1} + 7, \quad a_0 = 0.$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n.$

Multiply the recurrence relation by x^n, and sum over all values $n \geq 1$.

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} 7x^n$$

$$\underbrace{\sum_{n=1}^{\infty} a_n x^n}_{f(x) - a_0} = 3 \underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{x f(x)} + 7x \underbrace{\sum_{n=0}^{\infty} x^n}_{\frac{1}{1-x}}$$

$$\Rightarrow f(x) - a_0 = 3x f(x) + \frac{7x}{1-x}.$$

$$f(x) - 3x f(x) = a_0 + \frac{7x}{1-x} = \frac{7x}{1-x} \qquad (a_0 = 0)$$

$$f(x) = \frac{7x}{(1-x)(1-3x)}$$

Partial fractions:

$$f(x) = \frac{-\frac{7}{2}}{1-x} + \frac{\frac{7}{2}}{1-3x} = \frac{7}{2}\left(\frac{1}{1-3x} - \frac{1}{1-x}\right)$$

$$= \frac{7}{2}\left(\sum_{n=0}^{\infty} (3x)^n - \sum_{n=0}^{\infty} x^n\right) = \frac{7}{2} \sum_{n=0}^{\infty} (3^n - 1) x^n$$

$$= \frac{7}{2}\left(\sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty} x^n\right) = \frac{7}{2}\sum_{n=0}^{\infty}(3^n - 1)x^n$$

$$\Rightarrow \quad a_n = \frac{7}{2}(3^n - 1)$$

Ex. $\quad a_{n+2} = a_{n+1} + a_n \quad , \quad a_0 = 0 \ , \ a_1 = 1$

$0, 1, 1, 2, 3, 5, 8, \ldots$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Multiply by x^n and sum over all n.

$$\sum_{n=0}^{\infty} a_{n+2}x^n = \sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \quad \underbrace{\sum_{n=2}^{\infty} a_n x^{n-2}}_{\frac{f(x)-a_0-a_1x}{x^2}} = \underbrace{\sum_{n=1}^{\infty} a_n x^{n-1}}_{\frac{f(x)-a_0}{x}} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{f(x)}$$

$a_0 = 0$
$a_1 = 1$

$$\Rightarrow \quad f(x) - x = x f(x) + x^2 f(x)$$

$$(1 - x - x^2) f(x) = x$$

$$f(x) = \frac{x}{1 - x - x^2}$$

Note $\quad 1 - x - x^2 = 0 \Rightarrow x^2 + x - 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Rightarrow 1 - x - x^2$$

$$= -(x^2 + x - 1)$$

$$= -\left[x - \left(\frac{-1+\sqrt{5}}{2}\right)\right]\left[x - \left(\frac{-1-\sqrt{5}}{2}\right)\right]$$

$$= -\left[\frac{2}{\cdots}x - 1\right]\left[\frac{2}{\cdots}x - 1\right] \quad = \left[1 - \frac{2}{\cdots}x\right]\left[1 - \frac{2}{\cdots}x\right]$$

$$= \frac{-\left[\frac{2}{-1+\sqrt{5}}x-1\right]\left[\frac{2}{-1-\sqrt{5}}x-1\right]}{\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{-1-\sqrt{5}}{2}\right)} = \frac{-\left[1-\frac{2}{-1+\sqrt{5}}x\right]\left[1-\frac{2}{-1-\sqrt{5}}x\right]}{\frac{1-5}{4}}$$

$$= \left[1-\frac{2}{-1+\sqrt{5}}x\right]\left[1-\frac{-2}{1+\sqrt{5}}x\right]$$

Then

$$f(x) = \frac{x}{1-x-x^2} = \frac{A}{1-\frac{2}{-1+\sqrt{5}}x} + \frac{B}{1-\frac{-2}{1+\sqrt{5}}x}$$

where

$$A\cdot\left(1-\frac{-2}{1+\sqrt{5}}x\right) + B\cdot\left(1-\frac{2}{-1+\sqrt{5}}x\right) = x$$

$$\Rightarrow \quad A+B=0 \Rightarrow A=-B$$

$$\frac{2}{1+\sqrt{5}}\cdot A - \frac{2}{-1+\sqrt{5}}B = 1$$

$$\frac{2}{1+\sqrt{5}}\cdot A + \frac{2}{-1+\sqrt{5}}A = 1$$

$$\left[2(-1+\sqrt{5}) + 2(1+\sqrt{5})\right]A = (1+\sqrt{5})(-1+\sqrt{5}) = 4$$

$$A = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}$$

Thus

$$f(x) = \frac{1}{\sqrt{5}}\cdot\frac{1}{1-\frac{2}{-1+\sqrt{5}}x} - \frac{1}{\sqrt{5}}\cdot\frac{1}{1-\frac{2}{1+\sqrt{5}}x}$$

$$= \frac{1}{\sqrt{5}}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{-1+\sqrt{5}}x\right)^n - \frac{1}{\sqrt{5}}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{1+\sqrt{5}}x\right)^n$$

$$= \sum^{\infty} \frac{1}{\sqrt{5}}\left[\left(\frac{2}{-1+\sqrt{5}}\right)^n - \left(\frac{2}{1+\sqrt{5}}\right)^n\right]x^n$$

$$= \sum_{n=0} \frac{1}{\sqrt{5}} \left[\left(\frac{2}{-1+\sqrt{5}} \right)^n - \left(\frac{2}{1+\sqrt{5}} \right)^n \right] x^n$$

$$\Rightarrow \quad a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{2}{-1+\sqrt{5}} \right)^n - \left(\frac{2}{1+\sqrt{5}} \right)^n \right]$$